Reply to "Comment on 'Stability and quality factor of a one-dimensional subwavelength cavity resonator containing a left-handed material' "

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The Comment of Li *et al.*[1] consists of two parts, and the first part is about the dispersion relation of one-dimensional (1D) subwavelength cavity proposed by Engheta.[2] The exact dispersion relation for the 1D subwavelength cavity has the form[1]

$$\frac{\tan(n_1 k d_1)}{\tan(n_2 k d_2)} = -\frac{n_1 \mu_2}{n_2 \mu_1}, \tag{1}$$

where k is the wave number in free space, d_1 and d_2 are the thicknesses of the conventional dielectric material (RHM) and left-handed material (LHM) layers, respectively, and μ_1 (μ_2) and n_1 (n_2) are the permeability and refraction index of the RHM (LHM), respectively. Since $\mu_2 < 0$ and $n_2 < 0$, Eq. (1) can be rewritten as $\tan(n_1 d_1)/\tan(|n_2| d_2) = n_1 |\mu_2| / |n_2| \mu_1$. For a given frequency, the values of k, n_1, $|n_2|$, μ_1, and $|\mu_2|$ are determined, and obviously there exist possible solutions of small d_1 and d_2, for which $n_1 k d_1 < \pi/2$ and $|n_2| k d_2 < \pi/2$, and d_1 decreases when d_2 is decreased. Thus, as indicated by Engheta, this dispersion relation does not show any constraint on the sum of thicknesses of d_1 and d_2, and the subwavelength cavity is available in principle. Using the small-argument approximation for the tangent functions in Eq. (1), Engheta further simplified the dispersion relation to

$$\frac{d_1}{d_2} \simeq \frac{|\mu_2|}{\mu_1}. \tag{2}$$

The two sides of Eq. (2) are completely equal when the RHM and LHM layers are perfectly matched in impedance and Eq. (2) can then be rewritten as $d_1/d_2 = |n_2|/n_1$. In this special situation, if a wave in the subwavelength cavity transverses one material layer (with refraction index n_i, where $i=1$ or 2) and enters another material layer, it then has a phase change of $n_i k d_i$ and the resonant condition of the cavity can be physically expressed as $2\Sigma_i n_i k d_i = 0$, leading to the dispersion relation of the cavity only depending on the thickness ratio d_1/d_2. However, in a general case where the two material layers are mismatched, the complete transmission of a wave from one layer into another includes multiple reflection-incidence events occurring in the first layer and thus the total transmitted wave in the second layer has an effective phase change for transversing the first layer that is not again linearly proportional to the layer thickness. As a result, the dispersion relation of the cavity depends on the specific thicknesses of two material layers in the cavity [as shown by Eq. (1)]. In other words, the dispersion relation depends on both the ratio and the sum of the two thicknesses.

Undoubtedly, for the general case, Eq. (2) may give a good estimate of the ratio of d_1 and d_2 for the design of a subwavelength cavity at a desired frequency, and the thinner the cavity is, the more accurate the estimate is. It is known that a LHM is inherently dispersive and lossy. This means that μ_2 is generally a function of frequency and the frequency may also be distinguished from the variation of the value of μ_2. However, it is still a problem if the relation (2) is certainly effective to represent the dispersion relation of the subwavelength cavity. From the Comment of Li *et al.*,[1] it seems that they do not realize this problem existing in the subwavelength cavity. To understand well the cavity as well as its properties, this problem must be clarified physically.

Let us first consider an ideal case of the LHM in the cavity, in which the LHM has a frequency region where the permeability μ_2 is a constant [but here the permittivity of the LHM (ϵ_2) need not be assumed so] and the resonant frequency lies in this region. In this situation, the resonant frequency of the cavity cannot be determined with the relation (2), even when it is able to give a good estimate of the ratio d_1/d_2. However, the dispersion relation (1) can still determine the resonant frequency for the ideal case because the frequency is explicitly included in it through the quantity k. But k vanishes in the derivation of the relation (2) from the relation (1). In this derivation, the tangent functions in Eq. (1) are expanded in Taylor series only to first order. Apparently, for the ideal case, some high-order terms of the expansion shouldn't be neglected, and when the tangent function is expanded to third order, the resulted equation becomes

$$\varsigma = \hat{\mu} + \frac{\alpha^3 \hat{\mu}^3 - \varsigma^3}{3(1+\varsigma)^2} n_1^2 k^2 d^2, \tag{3}$$

where $\varsigma = d_1/d_2$, $\hat{\mu} = |\mu_2|/\mu_1$, and $\alpha = \sqrt[3]{|\varepsilon_2 | \mu_1 / \varepsilon_1 |\mu_2|}$. Equation (3) corresponds to Eq. (4) in our previous paper[3] and it explicitly contains the quantity k as well as the total thickness of the material layers $d = d_1 + d_2$. In the ideal case, it is clear from Eq. (3) that the dispersion relation for the subwavelength cavity depends as closely on the cavity thickness d as on the thickness ratio ς. But this does not contradict what the cavity thickness is optional for the subwavelength cavity with a certain frequency, as demonstrated in our previous paper. Compared to the relation (3), the total thickness effect is completely omitted in the relation (2) and thus it fails to determine the resonant frequency for the ideal case, i.e., it fails to represent the dispersion relation. Evidently, when the LHM in the cavity has a frequency region of interest with small permeability dispersion, i.e., μ_2 only has a weak dependence on the frequency, the relation (2) might give a wrong prediction on the resonant frequency and thus proves not to be an effective expression for the dispersion relation. To show this, we assume that in a frequency region of interest the permeability of the LHM has a form of $\mu_2 \approx -\mu_1(a+bk)$, where b reflects the permeability dispersion over the frequency region, and we rewrite Eq. (3) as

$$\varsigma = a + bk + \frac{\alpha^3 \hat{\mu}^3 - \varsigma^3}{3(1+\varsigma)^2} n_1^2 k^2 d^2. \tag{4}$$

When the dispersion parameter b is so small that the second term on the right side of Eq. (4) is less than the last term ($\alpha \neq 1$ for the general case), the dispersion relation of the thin cavity does not again mainly depends on the thickness ratio ς. In this case, the relation (2) is also not enough for the design of the subwavelength cavity, even if it is able to provide a good estimate of the thickness ratio and the total thickness of the cavity must be specified in order to make the cavity operating at the desired frequency.

Of course, for a nonzero permeability dispersion of the LHM, the last term in Eq. (4) would be far smaller than the term bk as long as the total thickness d is decreased so small and the relation (2) then becomes an effective expression for the dispersion relation. However, as the LHM is an artificial material, the thickness of the LHM layer is actually limited by the size of the unit cell of the material and so is the thickness of the compact cavity. In a practical case, the relation (2) even possibly fails to give a good estimate for the thickness ratio. For example, if the LHM made in Ref. 4 is used to construct a one-dimensional compact cavity, the minimal thickness of the LHM layer, which only has a single layer of unit cells, is equal to 5 mm. The LHM with negative permittivity and permeability operates at frequencies around 9 GHz. Assuming that the RHM in the cavity is air and we take the following parameters for the LHM layer: $\mu_2 = -1.5$, $\epsilon_2 = -1$, and $d_2 = 5$ mm. From the relation (2), the thickness of the air layer in the cavity is found to be $d_1 = 7.5$ mm. However, from the relation (1), we find the exact thickness of the air layer to be $d_1 = 5.7$ mm for the operation frequency $f = 9$ GHz. Therefore, for a practical case, the relation (2) might be not as accurate as expected and evidently the total thickness of the cavity becomes an important effect that must be included in the dispersion relation.

Even if the thickness of the cavity is allowed to be small arbitrarily and the last term in Eq. (4) may be completely neglected in mathematics, the relation (2) still losses some important effects in physics. In the relation (2), the permittivity of the LHM (ϵ_2) is not included at all but in the relation (1) it is well included through the refraction index n_2. Thus, even for the case of very thin cavity, the loss of the LHM from the permittivity ϵ_2 cannot be seen to have any influence on the quality factor of the cavity from the relation (2), though it seems to be very accurate. On the other hand, the influence of the permeability loss on the quality factor cannot be found out from the relation (2) too, as the ratio of d_1/d_2 is always a real number. And it is more impossible to find out the effect of the cavity compactness on the quality factor, which was demonstrated in our previous paper. As the exact dispersion relation (1) is a transcendental equation, it is necessary to find its effective approximation for studying analytically the properties of the cavity and hence the relation (2) needs to be modified to include some important physical effects. Here, we would like to point out once more that as the dispersion relation itself, the relation (2) still needs to be revised for some theoretical or actual cases, as discussed above. We wish this reply could clarify the questions on the dispersion relation of the subwavelength cavity.

The second part of the Comment of Li *et al.* is about the resonant frequency tolerance and this part is meaningful. In the analysis of this problem, the dispersion of the LHM layer in the cavity should be considered, except for the case in which it is small and the two material layers in the cavity are mismatched.

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[1] J. J. Li, L. Thylen, and S.-Y. Wang, preceding paper, Phys. Rev. B **79**, 207101 (2009).
[2] N. Engheta, IEEE Antennas Wireless Propag. Lett. **1**, 10 (2002).
[3] L. F. Shen, S. He, and S. H. Xiao, Phys. Rev. B **69**, 115111 (2004).
[4] L. Yan, L. X. Ran, H. S. Chen, J. T. Huangfu, X. M. Zhang, K. S. Chen, T. M. Grzegorczyk, and J. A. Kong, IEEE Trans. Microwave Theory Tech. **53**, 1522 (2005).